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NON-EUCLIDEAN GEOMETRY IN THE TRAINING OF MATHEMATICS TEACHERS[1]

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Suspend a weight by a string, tie the same kind of string below the weight, and pull the lower string. What happens? This little lesson on inertia carries a message over and beyond the physics involved. It is a good object lesson in personal relations. Pull gently (persuasively) and the weight cooperates and pulls with you. Jerk it sternly and it pulls against you. Inertia is apt to imply inaction or stagnation to the non-scientist. Curriculum considerations should, however, take into account both kinds of inertia. There is an inertia of no change and an equally potent and dangerous inertia of change for the sake of change.

If the undergraduate mathematics program suffers from inertia, it is of the first kind. With two significant exceptions, the program has remained static for a surprisingly long time. Those two exceptions have been forced upon the college to a large extent by the high schools. They are the introduction of sub-college courses for the benefit of students who desire college mathematics but find themselves deficient in college prerequisites; and the recent trend toward providing courses for general education at the freshman level. Otherwise, not only as to courses offered, but as to content "present day college mathematics was established some two hundred years ago" (Scherk and Kwizak, 1951). There is little wonder that mathematics teachers are pitied or envied, depending on the temperament of the other party, because "mathematics is always the same." It seems to be a popular impression that mathematics in its present complete (?) state of development, if not created during the first six days was handed down from Mt. Sinai.

Perhaps the undergraduate mathematical offering is just as it should be; however, the weight of tradition may serve as a potent drug to deaden our sensitivity to needed or advantageous change. Then it behooves us to make sure from time to time that the path of tradition is the path we should be following—or do something about it.

Those who specialize in college mathematics may be roughly classified into three groups, though the classifications are not mutually exclusive. There are those who will apply mathematics: the engineer, actuary, statistician. A second group consists of those who will create mathematics, the research mathematician. The third group consists of those who will popularize, and disseminate mathematical knowledge, the teachers of mathematics.

[1]Based on a paper presented at the 1951 meeting of The Tennessee Academy of Science, Austin Peay State College, Clarksville, Tennessee.

It is not within the scope of this paper to attempt even a partial evaluation of the mathematics curriculum as it bears upon the training of the first two of these groups. However, it can be remarked parenthetically that the gap between the undergraduate program in mathematics and the frontiers of research constitute a veritable chasm. It has been estimated that more mathematics has been created since 1900 than was created prior to that time. Yet practically none of it has found its way beyond the upper reaches of graduate instruction. If the thought that current and recent research will ultimately make itself felt at the undergraduate level seems fantastic, let us remember that the testimony of history argues eloquently in favor of the idea. Scherk and Kwizak have expressed the opinion that the downward movement of the products of the research of this mathematical era is overdue (1951).

The total training programs for the three groups of students will of course differ. It is highly doubtful that the undergraduate mathematics program should be identical for all three groups. Then let us consider the undergraduate mathematics offering from the standpoint of the teacher of mathematics. It is a pedagogical truism that the teacher should know his subject well beyond, both vertically and horizontally, the point he is to teach. On this basis alone, a good case can be made for everything usually found in the undergraduate mathematics program. A knowledge of algebra makes for a better arithmetic teacher; the study of differential equations makes elementary calculus more meaningful. And so it goes.

Those of us whose primary concern lies with the training of teachers have more cause to look critically at our offerings and look more frequently than do those whose primary concern is with training research workers or applied mathematicians. Marked changes have taken place and continue to take place in the educational objectives of the American secondary school. The subject matter preparation of teachers should be adjusted to these changes. The now current double track plan for high school mathematics is a case in point. This idea, which is basically sound and consistent with prevalent educational objectives has not met with too much favor or enthusiasm on the part of high school teachers. This is quite understandable because their training has not equipped them to teach the newer type of courses found in track two. It is quite possible that this situation calls for a different kind of college mathematics.

But let us examine the training of the teacher of high school geometry in terms of preparation for teaching geometry. He may have had a course in solid geometry; analytic geometry, plane and possibly solid; a course which might be characterized as an extension of high school plane geometry, usually called college geometry; and a course in synthetic projective geometry. This isn't the minimum, nor is it the average, it is very nearly the maximum for those without graduate work.

E. P. Lane (1930) has outlined the steps in setting up a geometry as follows: (1) Select the space. (2) Select the element. (3) Build the configurations. (4) Select the transformations. (5) Study the invariants. This classification omits one crucial step, namely, select the postulates to be used.

Geometries may be classified a number of ways: in terms of their spaces, plane, solid, or n-dimensional; in terms of the transformations, metric, projective, affine; in terms of technique of investigation, synthetic or analytic. Or they may be classified in terms of the postulational system employed, Euclidean or non-Euclidean, Archimedian or non-Archimedian. Our hypothetical teacher has had contact with spaces, or at least a space, other than two dimensional. This is admittedly meager when we think of abstract n-dimensional space of Reimannian geometry. Yet, it does afford the opportunity to observe that the validity of a statement will depend on the space under consideration, for instance the possible number of perpendiculars from a point on a line. He has had experience with both the analytic and synthetic techniques. He has had occasion to observe the significance of the transformations employed—the fact that the more general the transformation the fewer the invariants—that all projective properties are metric properties but the reverse does not hold. He has had opportunity to learn that the geometry of the high school is not a "finished story." Yet he has had no opportunity to observe the full significance of the role played by his system of postulates. Present day educational thought lays great stress on demonstrative geometry in terms of its value in providing opportunity for training in critical thinking. The youngster is supposed to learn the difference between truth and validity. He is supposed to grasp the full significance of the idea that his conclusions depend upon his basic assumptions. A speaker was recently heard to state that "Anybody with a lick of sense knows that the Pythagorean theorem is true." He evidently thought that the role of proof in demonstrative geometry is merely to convince one that the stated conclusions are correct. The assertion that the truth of our conclusions depends upon our assumptions is not very convincing when the possibility of other assumptions is ignored. Too frequently we admit that our conclusions depend on our assumptions but with the mental reservation that anyone who accepts assumptions other than our own are fools. It is quite interesting to watch the reactions of students in a non-Euclidean geometry class. The typical early reaction is something like this: "This is all a senseless game of logic but I shall try to remember the rules." They go along with the "absurdity" that it is absurd for a triangle to have an angle sum of two right angles. Then as progress is made and they begin to be able to see the whole panorama of the three geometries—hyperbolic, parabolic, and elliptic with the middle one being the limiting case, so to speak, of each of the others, there is change of attitude. The fifth postulate is no longer a statement of absolute truth which unfortunately cannot be proved but becomes merely one of three alternatives, any one of which is equally palatable to our intuitive sense of things as they actually exist. The study of geometry can make one more tolerant of the other fellow's point of view, but it can very well have the opposite effect if the student is left with the conviction that his conclusions are the only acceptable ones which could be reached. Exercises in reaching ridiculous conclusions from ridiculous assumptions will not produce the desired effect.

It is a part of the rightful heritage of everyone to know something of the history of the development of mathematical thought and some conception of what mathematics really is. There is no better way to give meaning to Russell's famous definition of mathematics as "the subject in which we never know what we are talking about nor whether what we are saying is true" than a course in non-Euclidean geometry. The high school teacher should have some direct contact with the epoch making step taken almost simultaneously in algebra and geometry only a little more than a hundred years ago, namely the rejection of some one or more of time honored postulates. While Hamilton's quaternions constituted an extension of the concept of number, the extension not conforming to previously accepted postulates, the rejection of the fifth postulate brought forth alternative parallel systems, equally valid and describing that part of the universe in which we move equally well. This provides opportunity in a most forceful way to point up the modern conception that mathematics is concerned with validity as opposed to the Kantian theory of mathematical truth. A valuable lesson is obtained in connection with establishing the consistency of the new geometries. We can show them to be as consistent as is Euclidean geometry or we can establish their consistency and that of Euclidean geometry itself in terms of real number. Yet we cannot establish the consistency of any of them in an absolute sense. Once again pointing up the tentativeness of all our conclusions. Perhaps the space of our existence has neither positive, zero, or negative curvature—for all we know it is a variable. Though the development is synthetic, brief excursions into analytic representation on the hyperbolic plane are quite profitable. Mere efforts at setting up a coordinate system, and the difficulties encountered, make it obvious why the assumption of space of zero curvature is the more practical. On the Euclidean plane it is immaterial whether we think of the point (X, Y) as being X units horizontally from the origin and then Y units vertically, or Y units vertically from the origin and the X units horizontally. This is not the case on the hyperbolic plane, the above operations will define two different points. We can think of the Euclidean plane as being mapped by lines perpendicular to the axes, as lines parallel to the axes, or as lines equidistant apart. Yet, on the hyperbolic plane, each of these three concepts are different, all of which are unsatisfactory in terms of setting up a coordinate system. There is opportunity here for a valuable lesson concerning the approximate nature of measurement. If we knew our universe were hyperbolic or elliptic the engineer would still use the Euclidean assumption because his results would still be as nearly exact as his limitations of measurement now permit. Thus, we cannot contradict the acute angle or obtuse angle assumption by actual measurement.

Such ideas as the foregoing are a valuable part of the equipment of the mathematics teacher. No contention is made that they can be obtained only through the study of non-Euclidean geometry. But a course of this kind does afford the opportunity for some insight into them at the most elementary level of advanced mathematics. The fact that we have the exact counterpart of high school plane geometry as to method and content is another argument for its inclusion in a teacher training program. Perhaps of all the sins of teachers, the one most difficult to avoid is failure to see the subject through the eyes of the student. Here the prospective teacher finds himself in almost identically the same situation, for him, which his geometry pupils face. It should make him more sensitive to the pupils' difficulties, and more capable of helping him through them.

Bibliography

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NEWS OF TENNESSEE SCIENCE

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which carries the gamma source into the center of the solid metal when not in use.

The University of Tennessee has contracted with the U. S. AEC for a fundamental geological investigation of the State's uranium-bearing Chattanooga shale. The project is to be carried out by the Department of Geology and Geography of the university, under the direction of Dr. Paris B. Stockdale as project leader. The work will include sample gathering, strata measurements and mapping, and extensive chemical analyses in association with the Department of Chemistry. Dr. Harry J. Klepser, professor of geology, Stuart Maher, and Ernest Russell will be associated with Dr. Stockdale on this project.

John H. Bailey, associate professor of biology at the East Tennessee State College, Johnson City, has been elected as the Tennessee representative of the National Biology Teachers Association for the conservation project supported by this organization.

Dr. A. J. Sharp, who has been serving as acting head of the Department of Botany at the University of Tennessee, has now been appointed permanent head of that department.

Dr. Donald R. Griffin, associate professor of zoology in Cornell University and a national lecturer for the Society of the Sigma Xi, has made two addresses in Tennessee on the subject of "Sensory Physiology and the Orientation of Animals." Dr. Griffin spoke before the Vanderbilt University chapter of Sigma Xi on January 24, and at the University of Tennessee on January 25.

On January 23, Mr. Richard C. Starr, formerly instructor in botany at Vanderbilt University, spoke to the Vanderbilt Science Club on his experiences during 1950-1951 as a Fulbright Scholar at the University of Cambridge, England. Mr. Starr recently completed the requirements for the Ph.D. degree at Vanderbilt, and is now an instructor in the Department of Botany, Indiana University, Bloomington, Indiana.

Recent Publications by Tennessee Authors

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