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THE QUADRIC OF WILCZYNSKI

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The purpose of this note is to state and prove a theorem concerning the quadric of Wilczynski. In the first place, let S be a non-ruled surface in ordinary projective space. If the asymptotic curves are chosen as parametric, then S is an integral surface of a pair of differential equations which can be written in the Fubini canonical form [1][1]

$$x_{uu} = px + \theta_u x_u + \beta x_v, \quad x_{vv} = qx + \gamma x_u + \theta_v x_v \qquad (\theta = \log \beta\gamma), \tag{1}$$

in which subscripts indicate partial differentiation, and the coefficients are functions of u,v. We select an ordinary point P_x of the surface S as one vertex of the usual local tetrahedron of reference whose vertices are the points x, x_u, x_v, x_{uv}.

Two lines $l_1(a,b)$, $l_2(a,b)$ are reciprocal lines at a point P_x of the surface if the line $l_1(a,b)$ joins the point P_x and the point y defined by

$$y = -ax_u - bx_v + x_{uv} \tag{2}$$

and the line $l_2(a,b)$ joins the points ρ, σ defined by placing

$$\rho = x_u - bx, \quad \sigma = x_v - ax, \tag{3}$$

where a,b are functions of u,v. It follows from equations (3) that

$$\begin{aligned} \rho_v &= -(b_v + ab)x - b\sigma + x_{uv}, \\ \sigma_u &= -(a_u + ab)x - a\rho + x_{uv} \end{aligned} \tag{4}$$

The lines $\rho\rho_v$ and $\sigma\sigma_u$ intersect the line $l_1(a,b)$ in the respective points

$$(ab - b_v)x + y. \quad (ab - a_u)x + y. \tag{5}$$

The harmonic conjugate of the point P_x with respect to these two points is the point whose local coordinates are

$$\begin{aligned} x_1 &= ab - \tfrac{1}{2}(a_u + b_v), \\ x_2 &= -a, \\ x_3 &= -b, \\ x_4 &= 1. \end{aligned} \tag{6}$$

[1]Lane, E. P. 1942. *A treatise on projective differential geometry.* Chicago.

We now propose to prove the following theorem: <u>At a point P_x on a surface the locus of the point (6) for a line $l_1(a,b)$ which is the cusp-axis of a variable pencil of conjugate nets on the surface is the quadric of Wilczynski.</u>

The curvilinear differential equation of any conjugate net N_λ on the surface S can be written in the form

$$dv^2 - \lambda^2 du^2 = 0 \qquad (\lambda \neq 0), \tag{7}$$

where λ is a function of u,v. Let us denote the two curves of the net N_λ that pass through the point P_x by C_λ and $C_{-\lambda}$ according as the direction dv/du has the value λ or $-\lambda$. The curvilinear differential equation of the pencil of conjugate nets determined by the net (7) is

$$dv^2 - \lambda^2 h^2 du^2 = 0 \qquad (h \neq 0), \tag{8}$$

where h is a constant. By a hypergeodesic is meant a curve C_λ which satisfies a differential equation of the form

$$\lambda' = A + B\lambda + C\lambda^2 + D\lambda^3 \qquad (\lambda' = \lambda_u + \lambda\lambda_v), \tag{9}$$

in which the coefficients A,B,C,D are functions of u,v. To each such family of hypergeodesics is associated a cusp-axis, which is the line $l_1(a,b)$ for which

$$a = \tfrac{1}{2}(\theta_v + C), \qquad b = \tfrac{1}{2}(\theta_u - B). \tag{10}$$

If equation (8) is solved for h, and if h is then eliminated by total differentiation with respect to u, it becomes apparent that the curves of a pencil of conjugate nets constitute a family of hypergeodesics for which

$$A = D = 0. \qquad B = \lambda_u/\lambda, \qquad C = \lambda_v/\lambda. \tag{11}$$

Moreover, the cusp-axis at the point P_x is the line $l_1(a,b)$ for which a and b are given by the formulas

$$a = \tfrac{1}{2}(\theta_v + \lambda v/\lambda), \quad b = \tfrac{1}{2}(\theta_u - \lambda u/\lambda). \tag{12}$$

With these expressions for a and b, the coordinates of the point defined by equations (6) become

$$\begin{aligned} x_1 &= \tfrac{1}{4}(\theta_v + \lambda v/\lambda)(\theta_u - \lambda u/\lambda) - \tfrac{1}{2}\theta_{uv}, \\ x_2 &= -\tfrac{1}{2}(\theta_v + \lambda v/\lambda), \\ x_3 &= -\tfrac{1}{2}(\theta_u - \lambda u/\lambda), \\ x_4 &= 1. \end{aligned} \tag{13}$$

Finally, homogeneous elimination of λ yields the equation of the quadric of Wilczynski

$$x_2x_3 - x_1x_4 - \tfrac{1}{2}\theta_{uv}x_4^2 = 0. \tag{14}$$