A TECHNIQUE FOR HEAT TRANSFER ANALYSIS BY ELECTRICAL ANALOG

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In the design of modern high performance heat transfer equipment, and indeed, in the mere measurement of temperature under many conditions, problems arise which can be solved only with difficulty. A typical problem is the flow of heat through a pin fin. Fins are used to improve performance of compact heat exchangers and to cool medium size electrical transformers as well as motors and hydraulic transmissions. A thermometer or thermometer well inserted in a fluid is a pin fin since heat can be conducted along the well to or from the wall. Under such conditions an analysis is often desirable to determine the probable accuracy of the thermometer reading.

For steady state operation the temperature gradient and heat transfer capacity of a pin fin may be analyzed by classical mathematics or by LaPlace transforms to yield the familiar equation:

$$\theta_{fin} - \theta_{bath} = C_1 e^{mx} + C_2 e^{-mx}$$

For solution, this may be simplified to:

$$\frac{\theta_{fin} - \theta_{bath}}{\theta_{base} - \theta_{bath}} = \frac{\cosh m(L-x)}{\cosh mL}$$

where $m = \sqrt{\frac{h\,p}{k\,A}}$ $\qquad L$ = length of fin

x = distance from base.

If the fin is uniformly tapered, a mathematical solution may be obtained by use of a modified Bessel Function of zero order:

$$\theta_{fin} - \theta_{bath} = C_1 I_0 B\sqrt{x} + C_2 K_0 B\sqrt{x}$$

$$\text{where } B = \sqrt{\frac{2\,h\,(\text{length of fin})}{k\,(\text{base thickness})}}$$

It will be noted that the above analyses are for steady-state conditions. If a time varying situation exists, the analysis is more difficult although solutions for many cases of unsteady-state heat conduction have been worked out in chart form by Gurney and Lurie, Boelter, and others.

A very useful and appealing tool for the analysis of this type of problem is the electrical analog model of the heat flow circuit. Such a circuit is usually a resistance-capacitance ladder network and falls short of the mathematical model of the physical situation only to the extent that it must be composed of discrete rather than infinitessimal modules.

Let us imagine a homogeneous plane wall subdivided into identical discrete layers of thickness Δx which have identical heat capacities. For analysis the mass of each layer may be treated as if it were concentrated at the centroid of the layer. The thermal capacitance of each layer is equal to its mass times its specific heat. In equation from this is $\rho A \Delta x c_p$ This is analogous to the capacitance in a reiterated R-C electrical ladder network.

The resistance to heat flow between adjoining layers is equal to the sum of the two half-resistances. That is, it is the resistance from one center-plane to the interface between layers plus the resistance from the interface to the next mid-plane. An exception to this rule is that at surface layers the total resistance equals one of the half-resistances plus the effect of the surface coefficient involved.

Calculation of thermal resistance is based on the basic Fourier equation:

$$q = - kA \frac{d\theta}{dx}$$

or for a finite interval Δx:

$$q = - kA \frac{\Delta\theta}{\Delta x}$$

Since this is analogous to Ohm's Law for flow of electric current: $i = \frac{E}{R}$

the half thermal resistance of a layer is $\frac{\Delta x}{2kA}$

Using the equation for heat transfer from a surface to a fluid:

$q = h_{c+r} A\Delta\theta$ the surface thermal resistance is $\frac{1}{h_{c+r} A}$

In the following table are listed the analogous heat flow and electrical equations placed in similar exponential forms. The electrical equations are those commonly found in elementary texts. Note particularly that when in this form the important analogous variables: voltage (E) and temperature (Θ); also current (i) and heat transfer rate (q) are in dimensionless form, thus eliminating the use of complex conversion factors in calculation.

TABLE
of
ANALOGOUS EQUATIONS

VARIABLE	HEAT FLOW	ELECTRICAL
Potential	$\frac{\theta - \theta_o}{\theta_s - \theta_o} = 1 - e^{\frac{-\alpha t}{\Delta x^2}}$	$\frac{E - E_o}{E_s - E_O} = 1 - e^{\frac{-t}{RC}}$
Rate of Energy Storage	$\frac{q_k}{\frac{kA}{\Delta x}(\theta_s - \theta_o)} = e^{\frac{-\alpha t}{\Delta x^2}}$	$\frac{i}{i_o} = e^{\frac{-t}{RC}}$
Energy Stored	$\frac{Q}{A\Delta x \rho c_p (\theta_s - \theta_o)} = 1 - e^{\frac{-\alpha t}{\Delta x^2}}$	$\frac{Q_E}{Q_\infty} = 1 - e^{\frac{-t}{RC}}$

It will be noted that the variable $e^{\frac{-t}{RC}}$ occurs in the electrical equations and that a similar expression $e^{\frac{-\alpha t}{\Delta x^2}}$ appears in the heat transfer equations. To compare the performance of single module resistance-capacitance electric circuits the "time constant" of a circuit is defined as that period of time which produces an exponent of minus one. If the capacitor in such a circuit is charged through a resistor by an applied voltage $E_s - E_o$, the attained voltage change, $E - E_o$, is 63.2% of the ultimate change when a time equal to the time constant has elapsed. The time constant of a simple electrical circuit is RC seconds and for an anlogous thermal circuit it is

$$\Delta x^2/\alpha . = \frac{\Delta x}{kA} \cdot (\Delta x A \rho c_p) = R_t C_t$$

, in hours. In application to an analogue circuit both time constants are computed for the same element of the section to be investigated and their ratio then applies to all data that may be recorded for the entire model.

The use of a resistance-capacitance ladder network to solve heat transfer problems is not new and many such devices have been used in the past. The most perplexing problem has been that the recording instruments used by other workers often drew enough energy from the circuit to disturb the normal current flow and voltage vs. time pattern. Compensation for this difficulty has taken two directions.

1—Provision of high voltage and very large capacitors so that high current flow and long time constants could be achieved. This solution is expensive and the high voltage and capacitance are dangerous to operators.

2—The use of very short time constants with readout being effected on oscilloscopes. Under these conditions readout may be awkward and of doubtful accuracy. In designing the heat transfer analog at Vanderbilt several steps were taken to reduce known objections of apparatus described in the literature. The design solutions were:

1—Use of mylar capacitors. This unit has a stable capacitance and will hold charge indefinitely. One micro-farad is used for each of the five modules in the analog.
2—The electrical time constant based on a single module is approximately ¼ second. This often gives valid charging or discharging histories running 10 to 20 seconds and nicely fits the readout used.
3—Readout is through a computer-type amplifier and then by a Brush Recorder which gives an immediate permanent record to a precision of one to two percent. The amplifier doe snot load the analog circuit. This is the key to the entire operation and will be described later. Banana plug patch cords are used to connect the amplifier to points in the circuit to be investigated.

Use of the apparatus in solving a problem is indicated in the figure. This illustrates how the problem of heat transfer through a tapered fin could be set up. The steps in the solution are:

1—The cross-section of the fin is first drawn to scale and then divided into five sections of equal area which therefore represent sections of equal heat capacities per unit length perpendicular to the cross-section. Each section then matches one of the electrical capacitors.
2—The thermal resistance from each boundary plane to the centroid of each section is computed and the corresponding potentiometer is set in the same proportion. Note that this requires the assumption that heat is conducted in a single direction. This is also a common assumption in the mathematical treatment of the problem.
3—One of the five elements is taken as the basis of time calculations. The temperature-time-constant of this element is computed as if the element were standing alone and heat flowed into it through the two preceeding half-resistances.
4—The electrical time constant of the analogous electrical element is also computed. This period of time on the readout graph is the period of heat model time calculated in step 3.
5—A "leak-off" potentiometer is adjusted to represent thermal resistance from the sides of each module to its surrounding fluid. These are set to the same scale used in setting the conduction resistance potentiometers.
6—The readout gives a voltage vs. time relationship for each section centerline when one patch cord is connected to ground.

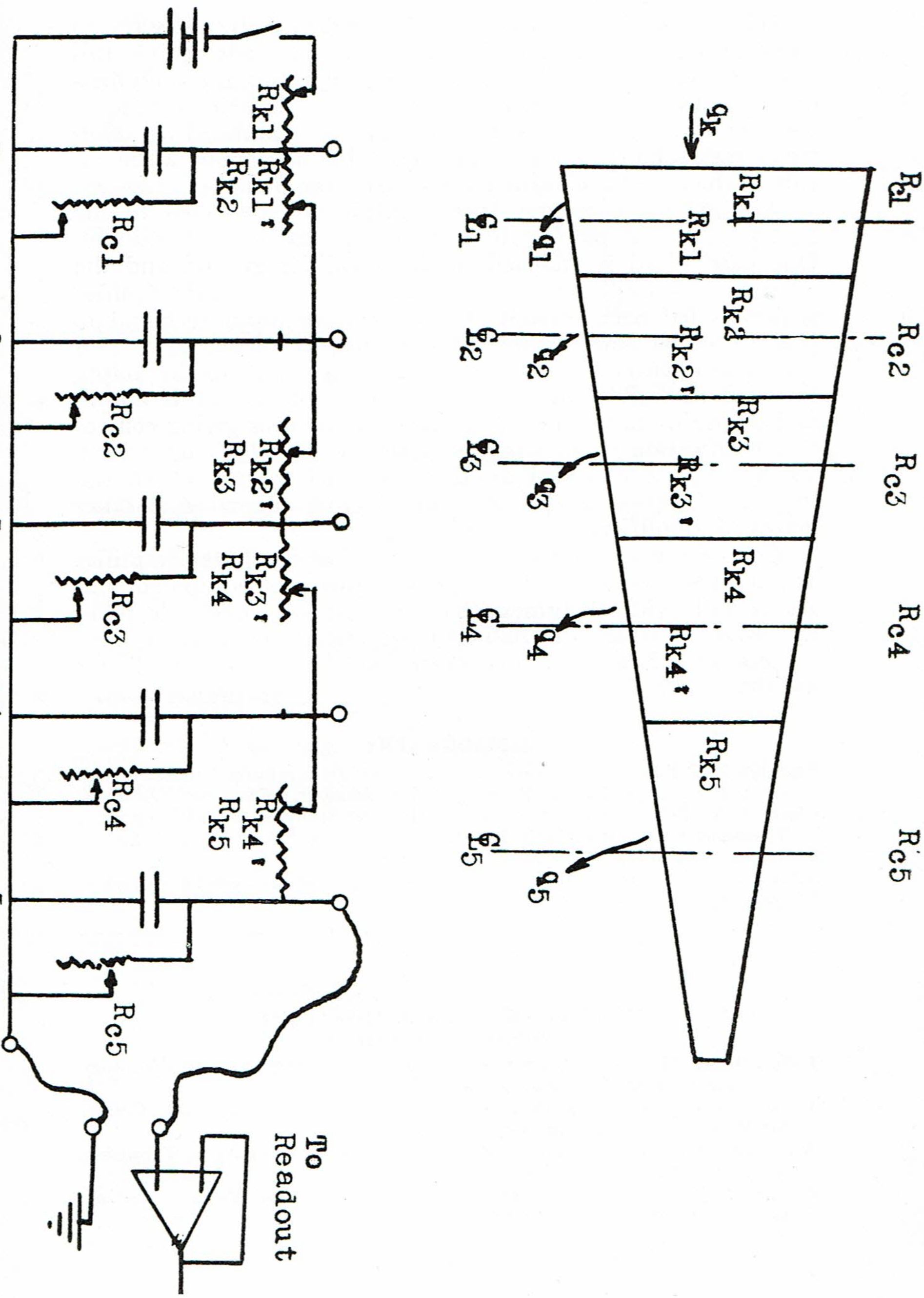
A steady-state analysis which can be made mathematically by Bessel Functions is made by the analog by allowing current to flow until steady voltages are obtained. We know of no mathematical analysis of an unsteady-state problem of this type other than by arithmetic means.

By arranging to feed in a programmed voltage vs. time relationship, an approximate unsteady-state analysis can be made for a wide variety of problems in addition to the one illustrated. These include a plane homogeneous wall, cylindrical sections and the operation of thermocouple probes in a medium in a medium of varying temperatures.

Because of the low voltages of zero to six volts, DC, which have so far been used, the analog lends itself to solution of several other problems. Circuits have been worked out that will simulate:

1—Heating and cooling of a slab or cylindrical solid with internal heat source and negative temperature coefficient of heat release.
2—Operation of a parallel or counter-flow heat exchanger.
3—Sudden contact of two materials of finite heat capacities and dissimilar temperatures.
4—Simple radiation problems.

The question arises as to the accuracy of the circuit. It can be tested for specific types of problems for which exact mathematical solutions exist. Verification against Boelter charts indicates a working agreement of two or three percent. Larger errors would naturally occur for such problems as the tapered fin. Nevertheless these errors are little more than the uncertainties in tabulated values of properties of materials.



qx
Rk1
Rk1'
Rk2
Rk2'
Rk3
Rk3'
Rk4
Rk4'
Rk5
Rc1
Rc2
Rc3
Rc4
Rc5
q1
q2
q3
q4
q5
1
2
3
4
5
To
Readout

Where surface coefficients are included it is necessary to treat them as constants. Valid complaints are made against this practice and it must be recognized that it limits the usefulness of any solution other than that of an arithmetic technique. Temperature varying coefficients can be simulated by electronic means but to do so will increase the complexity of the circuit, a chief virtue of which is its basic simplicity.

A Philbrick computer type amplifier is connected to the analog circuit by patch cords, producing flexibility in readout. One patch cord is attached to the amplifier ground and the other to the amplifier positive input terminal. The amplifier output is fed back directly to the negative input terminal to produce power amplification at a one-to-one voltage ratio. This results in essentially an infinite input impedance to the amplifier and negligible output impedance. The output is amplified in the usual manner by a second unit, thus giving control of amplification in driving the recorder. Karplus's text: Analog Simulation describes another feed-back circuit which has the same function as this one but which is adaptable to other makes of amplifiers.

Cost of this equipment is moderate. The computer amplifier and recorder are, of course, expensive pieces of equipment but are of such wide usefulness that they are very desirable tools. Exclusive of labor, the basic analog circuit using mylar condensers and 2 watt carbon potentiometers can be built for less than $100.

BIBLIOGRAPHY

Paschkis, V. & Baker, M. D. 1942. A Method for Determining Unsteady State Heat Transfer by Means of an Electrical Analogy. *Trans. ASME, v.* 64.

Clark, A. V. 1957. A Simplified Method for the Study of Two-Dimensional Transient Heat Flow Using Resistance Paper. *ASME paper* 57-8-9.

Schneider, P. J. 1955. *Conduction Heat Transfer.*

Kreith, F. 1958. *Principles of Heat Transfer, pp.* 118-120, 158-159.

Karplus, V. 1958. *Analog Simulation.*

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GLOSSARY

A	Cross section area
c_p	Specific heat
C	Electrical capacitance; constants
e	Napierian logarithm base
E	Voltage
$h_{c/r}$	Surface coefficient of heat transfer, convection plus radiation.
i	Electric current
$I_0(B\sqrt{x})$	Modified zero order Bessel Function of 1st. kind
$K_0(B\sqrt{x})$	Modified zero order Bessel Function of 2nd. kind.
k	Thermal conductivity
p	Perimeter
$q = \frac{dQ}{dt}$	Time rate of heat transfer
Q	Heat transferred
Q_E	Electrical energy transferred
R, R_t	Electrical resistance; Thermal resistance
t	Time
x	Length of heat flow path
Δx	Thickness of discrete section of thermal model
$\alpha = \frac{k}{\rho c_p}$	Thermal diffusivity
ρ	Density
θ	Temperature

SUBSCRIPTS

o	Original
∞	At infinite time
s	Applied