JOURNAL OF THE TENNESSEE ACADEMY OF SCIENCE
Volume 37, Number 2, April, 1962

A "BOHR-SOMMERFELD" NUCLEUS

Ray Hefferlin and Don Hall*
Southern Missionary College, Collegedale, Tennessee

Abstract

The Bohr quantum condition is applied to a particle moving in a deep square well to obtain an energy level diagram, which shows some qualitative agreement with experimental facts but is most interesting because of the analogy to the Bohr atom.

It is often enlightening in the field of physics to perform very rough calculations or carry through simple, even naive, derivations which are not expected to give the really correct answer at all. This paper falls in that category — it aims to present a primitive description of a nucleus by analogy to the familiar Bohr model of the atom.

We may begin by taking one nucleon and considering what it might do under the influence of the rest of the nucleus. It would be in general agreement with much experimental data to take the nuclear forces as being of short range, but approximately constant and very strong within that range. More specifically, we shall represent them by a "square potential well" with sides of height V_0. Presumably the ground state would be the situation in which the nucleon is at rest, "sitting on the bottom" of the potential well. Following custom, we shall assign zero potential energy to the region outside the nucleus; thus the particle in the ground state will have potential energy $-V_0$, kinetic energy 0, and total energy $-V_0$.

Now what happens if we give this nucleon some energy? One simple case that might be imagined is that of the nucleon rolling in a circular orbit around the edge of the region of negative potential, being restrained by the very high potential barrier. If the radius of this region is a, and the mass and speed of the particle are m and v, then the particle will have angular momentum $p_\phi = mva$. If we apply the Bohr quantum condition (Bohm, 1951) and note that p_ϕ is a constant, we obtain

$$\int p_\phi d\phi = 2\pi p_\phi = 2\pi mva = kh, \qquad (1)$$

where k is an integer and h is Planck's constant. The kinetic energy of the nucleon can then be written

$$T = \frac{p_\phi^2}{2ma^2} = \frac{(kh/2\pi)^2}{2ma^2} = \frac{h^2}{8\pi^2 ma^2}k^2. \qquad (2)$$

Since the nucleon is always inside the potential well, its total energy is its kinetic energy less V_0, and equation (2) states that this total energy can take on only certain values and that these values are quadratic in the "azimuthal quantum number" k. That "less V_0" is not really important for our purposes; we assume it is large enough that we do not get "above" the well, and we care more about how far above the ground state we are than how close to "ionization."

In another simple case the nucleon might be imagined as bouncing back and forth through the center of the nucleus. This time the Bohr quantum condition is applied to linear momentum:

$$\int p_r dr = mv \int |dr| = 4mva = \rho h. \qquad (3)$$

Here ρ is the "radial quantum number" and the integral is interpreted to mean "start at any point and keep going until back to the same point and going in the same direction." This gives a kinetic energy

$$T = \frac{P_r^2}{2m} = \frac{(\rho h/4a)^2}{2m} = \frac{h^2}{32ma^2}\rho^2, \qquad (4)$$

which is again proportional to the square of a quantum number. In contrast, the total energy $W = T - V$ (V measured *downward* from empty space) of the Bohr atmo is (White, 1934, pg 27)

$$-\frac{2\pi^2 me^4 Z^2}{h^2}\frac{1}{n^2}.$$

Now we are ready to consider orbits which have both angular and radial momentum — the Sommerfeld generalization. These orbits will look like those of a billiard ball on a round table — some regular polygons, some "star-shaped," some closed on themselves and some not. Each of these orbits may be characterized by its "impact parameter" p, as shown in Figure 1. If ϕ is measured from the point of closest approach B, then for any time between the last bounce at A and the next one at C, the distance of the nucleon from the center is given by $r = p/\cos\phi$. The constant linear momentum mv along the line AC can be resolved into two components:

$$p_r = mv \sin\phi$$
$$p_\phi = mv\,(p/\cos\phi)\cos\phi = mvp. \qquad (5)$$

Bohr's quantum condition may now be applied to each component. The azimuthal integral is easy because p_ϕ is again constant:

$$\int p_\phi d\phi = 2\pi p_\phi = 2\pi mvp = kh. \qquad (6)$$

The radial integral requires more thought. We begin by integrating just between the bounces at A and C:

*This work done under a National Science Foundation Undergraduate Research Participantship

$$\int_A^C p_r dr = \int_{-\arccos(p/a)}^{+\arccos(p/a)} (mv\sin\phi)\frac{p\sin\phi}{\cos^2\phi}d\phi = mvp\int_{-\arccos(p/a)}^{+\arccos(p/a)} \tan^2\phi\, d\phi$$

$$= mvp\left[\tan\phi - \phi\right]_{-\arccos(p/a)}^{+\arccos(p/a)} = 2mvp\left\{\frac{[1-(p/a)^2]^{1/2}}{p/a} - \arccos(p/a)\right\}. \qquad (7)$$

If the nucleon were following a simple closed path the meaning of $\int p_r dr$ would be quite clear. In any case it seems reasonable to multiply $\int_A^C p_r dr$ by $2\pi/[2\arccos(p/a)]$, giving

$$\int p_r dr = 2\pi mvp\left\{\frac{[1-(p/a)^2]^{1/2}}{(p/a)\arccos(p/a)} - 1\right\} = \rho h. \qquad (8)$$

In defense of this procedure we may note that equation (6) becomes (1) if $p = a$ and vanishes if $p = 0$. Furthermore, equation (8) becomes (3) when $p = 0$ and elementary calculus shows that its limit as p approaches a is zero. We may obtain a useful result by dividing (8) by (6):

$$\frac{\rho}{k} = \frac{[1-(p/a)^2]^{1/2}}{(p/a)\arccos(p/a)} - 1 \equiv f(p/a). \qquad (9)$$

The function f is graphed in Figure 2 over the interval [0, 1]; its range is $[0,\infty)$ and it has a unique inverse $f^{-1}(\rho/k)$.

The energy associated with each orbit may now be found as follows:

$$T = \frac{P_r^2}{2m} + \frac{P_\phi^2}{2mr^2} = \frac{P_\phi^2}{2mr^2}\left\{\frac{r^2p_r^2}{P_\phi^2} + 1\right\}$$

Substituting (5),

$$T = \frac{P_\phi^2}{2mr^2}\left\{\frac{\sin^2\phi}{\cos^2\phi} + 1\right\} = \frac{P_\phi^2\sec^2\phi}{2mr^2}$$

But $r = p\sec\phi = p\sec\phi$, so that, using (6) and following White (35, 46)

$$T = \frac{P_\phi^2\sec^2\phi}{2mp^2\sec^2\phi} = \frac{k^2h^2}{8\pi^2mp^2}. \qquad (10)$$

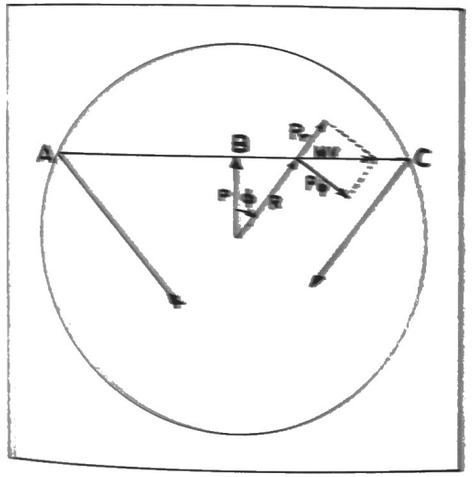

Figure 1. Path of excited nucleon in spherical region of highly negative potential, showing resolution of momentum and definitions of impact parameter p and azimuthal angle ϕ.

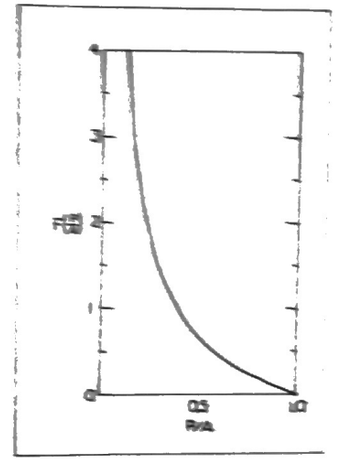

Figure 2. Graph of the function f defined by equation (9).

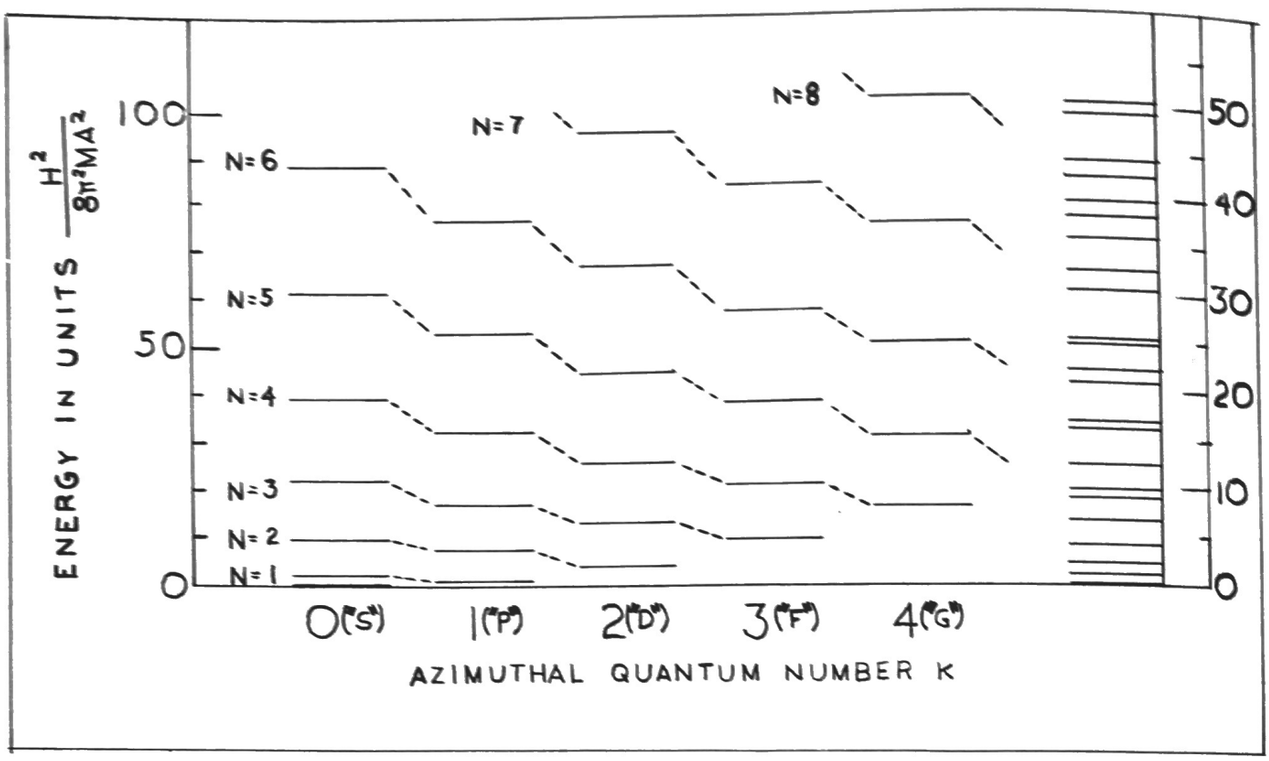

Figure 3. Energy level diagram for the Bohr nucleus. Levels with the same angular momentum ($kh/2\pi$) are in the same vertical column; levels with the same $n=k+\rho$ are connected with broken lines. There are additional levels above and to the right of those shown. The extreme right-hand column includes all states, regardless of k, with energy less than 50 $h^2/8\pi^2ma^2$. Notice the expanded scale for this column.

At first glance this seems to depend only on k, but we must remember that the possible values for p are determined by ρ *and* k through (9).

An energy level diagram may be constructed by computing $T = (k^2h^2/8\pi^2ma^2)\ [f^{-1}(\rho/k)]^{-2}$, where f^{-1} is found from Figure 2 for each pair of integer values of ρ and k. Such a diagram is shown in Figure 3. For the sake of the analogy to the atomic case, a "total quantum number" $n \equiv k + \rho$ is introduced and all levels with the same value of n are connected. There are several interesting points to note about this diagram:

1. As either k or ρ becomes larger while the other remains constant, the levels are spaced farther apart, rather than approaching a limit as in the Bohr atom.

2. The density of levels can be shown to be roughly $18ma^2/h^2$ for large values of T. Note that this is a constant, in contrast to the way the level density increases without limit as the energy approaches the ionization potential in the Bohr atom.

3. Levels with the same n do not tend to coincide for large values of n as they do in the Bohr atom. In fact, they range between $(1/\pi^2)\ (h^2n^2/8ma^2)$ for $k = n$, $\rho = 0$, and $(1/4)\ (h^2n^2/8ma^2)$ for $k = 0$, $\rho = n$.

4. The appearance of the diagram is the same for all nuclei, although one would have more faith in the basic assumptions of this model for larger nuclei. The only difference from one nucleus to another is in the scale on the energy axis, where the unit of energy used is inversely proportional to the 2/3 power of the atomic weight. In fact, using $r_0 = 1.2 \times 10^{-13}$ cm, this unit is $2.32 \times 10^{-5}\ A^{-2/3}$ erg or $14.5\ A^{-2/3}$ Mev. This would be 1 Mev for $A = 55$, or Mn.

In conclusion, let us consider a person who had some information on observed nuclear levels, but no theory except this one with which to interpret them. He would notice that this theory did well in predicting levels with separations of the order of an Mev and spaced somewhat irregularly, but that it completely failed to predict any of the details correctly. Thus we might expect him to decide that there was a grain of truth in the idea of a particle moving in a square well, but that it needed considerable improvement. He might even be led to invent wave mechanics!

LITERATURE CITED

Bohm, D. 1951. *Quantum Theory*. Prentice Hall, New York. 41.
White, H. E. 1934. *Introduction to Atomic Spectra*. McGraw-Hill, New York. 27.
Ibid. 45, 46.