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THE NUMBER OF PLANES CONTAINED IN THE COMPLEMENT OF A QUADRIC IN AN AFFINE GALOIS SPACE

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1. *Introduction.* Let q denote a power of an *odd* prime $p, q=p^n$, and let S_m denote the vector space of dimension m over the Galois field, $F=GF(q)$. It will be shown elsewhere that in S_m, every quadric meets every hyperplane, provided $m>3$. By a quadric, we mean a central, nondegenerate quadric; that is, a surface generated by equating a quadratic form of S_m of determinant $\neq 0$ to an element of F.

The number of planes are determined which have no point in common with a given quadric of S_3 (see Theorem 2). The argument is purely algebraic in nature. The only tools required are some results on squares and sums of squares in a Galois field as used by Dickson (1901).

2. *Preliminaries.* Let P denote the prime field GF(p) and let Ψ denote the Legendre symbol in F; that is, for elements c in $F, \Psi(c)=1, -1$, or 0, according as c is a nonzero square, a non-square, or is the zero element of F.

Lemma 1 (cf. Dickson [1,§62]).[1] *If* c *is a non-square of* P, *then* c *is a square in* F *if and only if* F *is of even degree* n *over* P.

Lemma 2 (Dickson [1,§64]). *Let* $\alpha, \alpha_1, \alpha_2$ *denote elements of* F, $\alpha_1\alpha_2 \neq 0$, *and let* $B(\alpha)$ *denote the number of solutions in* F *of* $\alpha=\alpha_1 x_1^2 + \alpha_2 x_2^2$. *Then* $B(\alpha) = q-\Psi(-\alpha_1\alpha_2)$ *or* $q+(q-1)\Psi(-\alpha_1\alpha_2)$ *according as* $\alpha\neq 0$ *or* $\alpha=0$.

In particular,

Corollary 1. $B(\alpha)>0$ for all α in F.

Lemma 3 (cf. Dickson [1,§168-169]). *Let* Δ *denote a given nonzero element of* F. *Every quadratic form* Q_m *of* S_m *with determinant* $\Delta\neq 0$ *can be reduced by a non-singular linear transformation of* S_m *to a form,* $\alpha_1 x_1^2+\dots+\alpha_m x_m^2$ $(\alpha_1\dots\alpha_m=\Delta)$; *moreover, in the case* m *odd,* Q_m is congruent to the form, $\Delta(x_1^2+\dots+x_m^2)$.

Remark 1. By Lemma 3, there is no loss of generality in assuming the given quadric of the paper to be of the form,

(1) $$a=x_1^2+x_2^2+x_3^2,$$

where a is an arbitrary element of F.

Convention. Let $b, \beta_1, \beta_2, \beta_3$ denote elements of F where at least one β_i is different from 0. In order to ensure that the planes of our discussion,

(2) $$b=\beta_1 x_1+\beta_2 x_2+\beta_3 x_3,$$

are distinct, it will always be assumed that the first nonzero β_i occurring in (2) has the value 1.

3. *The main results.* We shall need some additional notation and terminology. Let $N_1(a)$ denote the total number of planes (2) containing no point in common with the quadric surface (1). Let $N_2(a)$ denote the number of *homogeneous* planes (that is, planes (2) in which $b=0$) without a point in common with (1). Equivalently, $N_2(a)$ may be defined as the number of 2— dimensional subspaces of S_3 contained in the complements of (1). Let $N_3(a)$ denote the number of *oblique* planes (planes (2) with $\beta_1\beta_2\beta_3\neq 0$) which contain no point of the quadric (1). Finally, we divide the spaces S_3 into two classes, C and C′, where C consists of all S_3 for which either $p\equiv 1 \pmod 4$ or $p\equiv 3 \pmod 4$ and n is even, while C′ consists of those S_3 with $p\equiv 3 \pmod 4$ and n odd.

Theorem 1. *Any plane of* S_3 *contained in the complement of the quadric* (1) *is necessarily homogeneous. In fact,*

(3) $$N_1(a)=N_2(a)=\begin{cases} q+1 & \textit{if } \Psi(-a)=-1, \\ 0 & (\textit{otherwise}). \end{cases}$$

Moreover, in case $\Psi(-a)=-1$,

(4) $$N_3(a)=\begin{cases} q-5 & \textit{if } S_3\epsilon C, \\ q+1 & \textit{if } S_3\epsilon C'. \end{cases}$$

Proof. We consider two cases, according as the plane (2) is or is not parallel to a coördinate axis.

Case 1 (Non-oblique planes). The plane

(5) $$b=x_1+\beta x_2,\ \beta\epsilon F,$$

has a point in common with the quadric (1) if and only if the equation,

(6) $$a-b^2=c x_2^2-2b\beta x_2+x_3^2, c=1+\beta^2,$$

has a solution (x_2,x_3) in F. The latter equation may be written

(7) $$\frac{ac-b^2}{c^2}=\left(x_2-\frac{b\beta}{c}\right)^2+\left(\frac{1}{c}\right)x_3^2 \text{ if } c\neq 0,$$

and

(8) $$a=b^2-2b\beta x_2+x_3^2 \text{ in case } c=0.$$

The equation in (7) is always solvable, by Corollary 1. It will be noted that the case $c=0$ implies that $\beta\neq 0$. Hence, if $b\neq 0$, (8) has a solution with any preassigned value for x_3. If $b=0$, (8) is insolvable if and only if a is a non-square of F.

Therefore, a necessary and sufficient condition that (5) have no point in common with (1) is that

(9) $$b=0, \Psi(a)=-1, \Psi(-1)=1, \beta^2=-1,$$

in which case β has exactly two values, the two square roots in F of —1. Letting $N_1^*(a)$ denote the number of non-oblique planes (2) and $N_2^*(a)$ the number of non-oblique, homogeneous planes (2) having no point in common with the quadric (1), one obtains

[1] Numbers in brackets refer to the bibliography.

$$(10)\quad N_1^*(a)=N_2^*(a)=\begin{cases} 6 & (\Psi(a)=-1,\Psi(-1)=1),\\ 0 & (\text{otherwise}).\end{cases}$$

Case 2 (Oblique planes). We consider the general oblique plane

$$(11)\qquad b=x_1+\beta_2 x_2+\beta_3 x_3 (\beta_2\beta_3\neq 0).$$

Eliminating X_1 between (1) and (11), it follows that (1) and (11) have a common point if and only if the equation

$$(12)\qquad a-b^2=(c_2 x_2^2+2\beta_2\beta_3 x_2 x_3+c_3 x_3^2)- 2b(\beta_2 x_2+\beta_3 x_3),$$

where $c_2=\beta_2^2+1, c_3=\beta_3^2+1$, is solvable in F. The quadratic form enclosed in parentheses in (12) has determinant $\Delta=4(\beta_2^2\beta_3^2-c_2c_3)$. If $\Delta\neq 0$, this form is congruent to a form, $\alpha_1 x_1^2+\alpha_2 x_2^2$ $(\alpha_1\alpha_2\neq 0)$, by Lemma 3. Hence by the corollary to Lemma 2, it follows as in Case 2 that (12) is solvable.

Suppose then that $\Delta=0$, in which case it results that $c_2\neq 0, c_3\neq 0$. A simple computation shows that (12) assumes the form,

(13)

$$\frac{ac_2-b_2}{c_2^2}=\left(X-\frac{b\beta_2}{c_2}\right)^2+\frac{2bx_3}{c_2\beta_3}\left(X= x_2+\frac{c_3x_3}{\beta_2\beta_3}\right)$$

If $b\neq 0$, then (13) can be solved with X assigned arbitrarily. Assume therefore that $b=0$. Then (13) is insolvable if and only if ac_2 is a non-square of F.

We have thus shown that (1) and (11) fail to have a common point if and only if $\Delta=b=0, \Psi(ac_2)=-1$, or what is the same, if and only if

$$(14)\qquad b=0, \Psi(-a)=-1, \beta_2^2+\beta_3^2=-1.$$

But by Lemma 2, the system

$$\beta_2^2+\beta_3^2=-1, \beta_2\neq 0, \beta_3\neq 0,$$

has $q-5$ or $q+1$ solutions according as -1 is a square or a nonsquare of F. Therefore, if $N_3^*(a)$ denotes the number of homogeneous, oblique planes contained in the complement of (1), we have

(15)

$$N_3(a)=N_3^*(a)=\begin{cases} q+1 & (\Psi(-a)=\Psi(-1)=-1)\\ q-5 & (\Psi(-a)=-1,\Psi(-1)=1)\\ 0 & (\text{otherwise}).\end{cases}$$

Since $N_1(a)=N_1^*(a)+N_3(a), N_2(a)=N_2^*(a)+N_3^*(a)$, (3) follows on the basis of (10) and (15). The conclusion (4) results from Lemma 1 and (15), on recalling that -1 is a square in P if and only if $p\equiv 1 \pmod 4$.

Corollary 1. *The only cases in which* $N_1(a)>0$, $N_3(a)=0$, *occur when* $q=5, a=\pm 2$.

Corollary 2. *The case* $q=3, a=1$ *is the only one in which the complement of* (1) *contains all oblique, homogeneous planes of* S_3.

By Lemma 3 we obtain from (3) our principal result:

Theorem 2. *Let* Q *denote a ternary quadratic form over* F *of determinant* $\Delta\neq 0$, *and let* α *be an arbitrary element of* F. *All planes of* S_3 *contained in the complement of the quadric,* $Q=\alpha$, *are necessarily homogeneous (that is, subspaces); the number* $N(\alpha)$ *of such planes is*

$$N(\alpha)=\begin{cases} q+1 & \textit{if } \Psi(-\alpha\Delta)=-1,\\ 0 & \textit{otherwise}.\end{cases}$$

LITERATURE CITED

1. Dickson, Leonard Eugene. 1901. *Linear Groups,* Leipzig.

THE EIGHTEENTH TENNESSEE SCIENCE TALENT SEARCH

and

THE TWENTY-FIRST TENNESSEE JUNIOR ACADEMY OF SCIENCE

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The 1963 programs of the Tennessee Science Talent Search and the Tennessee Junior Academy of Science have given great impetus to the development of Junior Scientists in the state. The two programs, sponsored by the Tennessee Academy of Science with the cooperation and assistance of the science and mathematics teachers of the secondary schools, are directed by James L. Major, Clarksville High School, Clarksville, and Myron S. McCay, University of Chattanooga, Chattanooga, respectively.

To qualify for competition in the Tennessee Science Talent Search, entrants must be high school seniors, must pass a comprehensive examination, and report on an original science project. This year 36 students in 24 schools in 16 communities were selected as state winners. The winners came from all regions of the state, a fact most gratifying to the Talent Search Committee, one of whose stated aims is to have all schools in Tennessee participate in the program. The Academy forwarded to a large number of colleges the names of talented science students to be considered for admission, scholarships, for assistantships in science laboratories and similar positions.

The success of a continuing and productive Talent Search depends, in a large measure, upon a supporting

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