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SOME SETS OF INTEGERS

ECKFORD COHEN
University of Tennessee
Knoxville, Tennessee

1. If n is a positive integer, one may place

$$n = \prod_{p|n} p^e \tag{1}$$

where p runs over the *distinct* prime divisors of n. The set S of integers (1) in which all occurring e are > 1, the so-called "square-full" integers, have been investigated, along with various generalizations and analogues, by a number of writers during the past three decades (cf.[3][1] and the bibliography listed there). However, it does not seem to be widely appreciated that S is just the set of those integers n which are representable as a product of a square by a cube.

In this note we show how the more general sets of integers discussed in [2] and [3] can be characterized in a similar way. Let a,b denote fixed positive integers; we shall call a factorization $n = d^a \delta^b$ of n, where d and δ are positive integers, an (a,b)-*factorization* of n. If, moreover, d and δ are relatively prime, we shall call the decomposition *unitary;* otherwise, it will be called *non-unitary*.

Consider the set of integers $e = e(t)$,

$$\begin{aligned} & a\,i + (a+b)t, \; 0 \leqq i \leqq b-1; \\ & b\,j + (a+b)t, \; 1 \leqq j \leqq a, \end{aligned} \tag{2}$$

for arbitrary integral values of t. The set $S_{a,b}$ of [2] is defined to be those n in (1) for which every occurring exponent e is a number of the set (2) with $t \geqslant 0$. The set $S^*_{a,b}$ is the set of all n in (1) such that each e is a number of (2) with $t = 0$.

Suppose in what follows that a and b are positive and relatively prime, $(a,b) = 1$. Lemma 2.1 of [3] asserts that the set of $a + b$ integers in (2) with $t = 0$ is a complete set of residues (mod$(a+b)$). It is not difficult to deduce from this fact that $S_{a,b}$ *is precisely the sequence of all integers* n *which admit of an* (a,b)-*factorization, while* $S^*_{a,b}$ *is the subset of those integers* n *of* $S_{a,b}$ *which admit only of unitary* (a,b)-*factorizations.*

To prove these statements, one makes use of the fact that (2) contains all of the integers without repetition, in connection with the hypotheses on a and b, and the Fundamental Theorem of Arithmetic. It suffices to show that if e is an integer in (2), then e can be represented in the form $ax + by$ with x,y non-negative integers if and only if $t \geqslant 0$, and that e has such a representation with x and y both positive if and only if $t > 0$. The details are similar to those of the proof of Lemma 2.1 in [3] and may be left to the reader.

Next we point out how the sets of integers discussed in [2] can be similarly characterized. With $(a,b) = 1$, let $T_{a,b}$ denote the set of all n in (1) such that each exponent e is divisible by either a or b, and let $T^*_{a,b}$ denote the set of all n for which each exponent e is divisible by a or b but not by ab. $T_{a,b}$ may evidently be characterized as the set of all n which admit of a unitary (a,b)-factorization, while $T^*_{a,b}$ is the subset of those n of $T_{a,b}$ which admit of a *unique* unitary (a,b)-factorization.

2. Let now $S'_{a,b}$ denote the subset of those n of $S_{a,b}$ which admit only of non-unitary (a,b)-factorizations. Also, let $S'_{a,b}(x)$ denote the enumerative functions of $S'_{a,b}$; that is, $S'_{a,b}(x)$ is the number of integers $n \leqq x$ cotnained in $S'^{a,b}$. Evidently, $S'^{a,b}(x) = S_{a,b}(x) - S^*_{a,b}(x)$ where $S_{a,b}(x)$ and $S^*_{a,b}(x)$ are the enumerative functions of $S_{a,b}$ and $S^*_{a,b}$ respectively. Application of the two theorems on $S_{a,b}(x)$ and $S^*_{a,b}(x)$ proved in [3] lead therefore to the following formula for $S'_{a,b}(x)$ (ab) $= 1$, $b > a > 1$, then for all real $x \geqslant 2$,

$$S'_{a,b}(x) = r(a,b)x^{1/a} + s(a,b)x^{1/b} + O\left(x^{\frac{1}{a+b}} \log x\right), \tag{3}$$

where

$$r(a,b) = \zeta(b/a)\left(1 - 1/\zeta(c/a)\right)/\zeta(b),$$

$$s(a,b) = \zeta(a/b)\left(1 - 1/\zeta(c/b)\right)/\zeta(a),$$

and $\zeta(s)$ is the zeta-function.

3. The discussion of $S_{a,b}(a)$ and $S^*_{a,b}(x)$ in [3] was based on the following identities,

$$\begin{aligned} \rho_{a,b}(n) &= \sum_{d^{ab}\delta = n} \mu(d)\tau_{a,b}(\delta), \\ \rho^*_{a,b}(n) &= \sum_{d^{a+b}\delta = n} \mu(d)\rho_{a,b}(\delta), \end{aligned} \tag{4}$$

where $\rho_{a,b}$ and $\rho^*_{a,b}$ are the characteristic functions of $S_{a,b}$ and $S^*_{a,b}$, respectively, μ is the Moebius function, and $\tau_{a,b}(n)$ denotes the number of (a,b)-factorizations of n, $(a,b) = 1$. These formulas were established on the basis of generating functions. Using the interpretation of $S_{a,b}$ and $S^*_{a,b}$ noted in §1, we indicate now how the relations in (4) can be established without the aid of generating functions.

Suppose that a given positive integer e has representations, $e = ax + by = ax' + by'$, $(a,b) = 1$. Then $y' - y \geqslant a$, because the relation, $a(x - x') + b(y - y') = 0$, is incompatible if $y' - y < a$. Moreover, $by' - by \geqslant ab$, and if one makes use of (1), it follows that the maximal ab-th power divisors of the respective b-th power factors in two distinct (a,b)-factorizations of n must be distinct. That is, there is one-to-one correspondence between the (a,b)-factorizations of n and the ab-th power divisors δ^{ab} of n such that the conjugate divisors n/δ^{ab} are contained in $S_{a,b}$. This may be restated in the form,

$$\tau_{a,b}(n) = \sum_{d\delta^{ab} = n} \rho_{a,b}(d). \tag{5}$$

It is even simpler to deduce that

$$\rho_{a,b}(n) = \sum_{d\,\delta^{a+b} = n} \rho^*_{a,b}(d); \tag{6}$$

one need only observe that an integer of $S_{a,b}$ has a unique decomposition into a product of a number of $S^*_{a,b}$ and an $(a+b)$-th power.

The formulas in (4) result immediately on applying to (5) and (6) the following inversion formula ([1, (6.9)]),

$$f(n) = \sum_{d\delta^k = n} g(d) \rightleftarrows g(n) = \sum_{d^k\delta = n} \mu(d) f(\delta),$$

valid for all positive integers k. An analogous "unitary" inversion formula was used in [2] in studying the distribution of the sets $T_{a,b}$ and $T^*_{a,b}$.

[1] Numbers in brackets refer to Literature Cited.

LITERATURE CITED

1. Eckford Cohen. 1959. A class of residue systems (mod r) and related arithmetical functions, I. A generalization of Moebius inversion. *Pacific Journal of Mathematics.* 9:13-23.
2. ———. 1961. Unitary products of arithmetical functions. *Acta Arithmetica.* 7:29-38.
3. ———. 1963. On the distribution of certain sequences of integers. *American Mathematical Monthly.* 70:516-521.

NEWS OF TENNESSEE SCIENCE

(Continued from Page 47)

The University of Tennessee Medical Units in Memphis has been awarded a $14,100 grant by the National Science Foundation in support of research being conducted by Dr. William B. Winborn of the Department of Anatomy. His project is entitled "Fine Structure of Tissues Engaged in Transport."

Dr. Mark M. Jones, Professor of Inorganic Chemistry at Vanderbilt University, is the author of a new textbook published by Prentice-Hall. The title is "Elementary Coordination Chemistry."

Dr. Jones is presently conducting research at Vanderilt for the Air Force Office of Scientific Research and for the U. S. Department of Health, Education, and Welfare. His research interests within the field of coordination chemistry include: (a) the effect of coordination on the reactivity of aromatic ligands, (b) the coordination chemistry of arsenic, (c) catalytic effects of coordination on organic and inorganic reactions, (d) the thermochemistry of neutral complexes, (e) methods of studying complexation equilibria in solution, and (f) biological aspects of coordination chemistry.

The University of Tennessee has received a $10,000 grant from the National Science Foundation to support research relating to tularemia, or "rabbit fever," and functions of the blood in building up immunity to the disease. Dr. John M. Woodward, professor of bacteriology at UT and director of the reseach project, said the disease, found in rabbit, squirrel and other rodents, can be contracted by humans. Dr. Woodward began his study of tularemia in 1946 while working for his Ph.D. degree and has continued it with the aid of two previous grants, one each from the Office of Naval Research and the National Institutes of Health.

James O. Andes, head of the Department of Plant Pathology at the University of Tennessee Agricultural Experiment Station since 1950, retired this year. Dr. Andes has been associated with the College of Agriculture in various capacities since his graduation from the university in 1921.

The University of Tennessee has announced the inauguration of a doctoral program in nuclear engineering. The new program is an expansion of graduate level programs leading to the master's degree in nuclear engineering and the Ph.D. in engineering science. Course work for the Ph.D. in nuclear engineering is concentrated in four main areas—reactor analysis and design, shielding, reactor stability and controls, and heat transfer and fluid flow.

The University of Tennessee has been awarded a $10,000 grant from the National Science Foundation for research on copper-antimony alloys under the direction of Dr. Charles R. Brooks, assistant professor in the Department of Chemical and Metallurgical Engineering. The purpose of the study is to determine how small additions of a third element affect the hardening characteristics of the alloy.

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