JOURNAL OF THE TENNESSEE ACADEMY OF SCIENCE
Volume 44, Number 4, October, 1969

SOME PROPERTIES OF CERTAIN HYPER-SOLIDS

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INTRODUCTION

We shall discuss some metric relationships which characterize certain hyper-solids of n dimensions and result in some well-known elementary formulas being recognized as special cases of much more generalized forms. For instance, the perimeter and area of an equilateral triangle, as well as the surface and volume of a regular tetrahedron are four of the specific cases which are obtained from a coverall relationship.

THE HYPER-CUBE:

The definition of this solid is obtained by following an iterative procedure. Start with a point, a O-dimensional entity. Apply straight line motion to it over a distance of a units. Then its trace forms a line segment, a 1-dimensional magnitude. By moving this line segment out of its 1-dimensional space in a direction perpendicular to that space over a units a square, a 2-dimensional element, is obtained. Similar motion of the square creates a cube, a 3-dimensional manifold. Likewise with the cube, a tesseract (the 4-dimensional hyper-cube) will be formed. Continue this process indefinitely and it will lead to a set of hyper-cubes of n dimensions, where $\epsilon \{0, 1, 2, \ldots\}$.

Relationships:

1. Number of boundary manifolds:

$$M_{n,i} = 2^{n-i}\binom{n}{i} \qquad (1)$$

where $\binom{a}{b} \overset{=}{\text{def.}} \frac{a!}{b!(a-b)!}$ and $\binom{a}{b} = 1$ whenever $b = 0$.

Since n means the dimension of the solid under consideration, and i represents the dimensionality of the boundary manifold in question, $i \geq n$.

Proot:

To obtain relationship (1), note first that, owing to the definition of the n-dimensional hyper-cube, the recurrence formulas

$$M_{n,i} = 2M_{n-1,i} + M_{n-1,i-1} \quad ; \quad \begin{cases} n > 0 \\ n \geq i \end{cases}$$

$$M_{n,o} = 2M_{n-1,o} = 2^n \quad ; \quad n > 0 \qquad (0)$$

$$M_{o,o} = 1$$

hold. These formulas enable one to construct Table I quite rapidly.

But

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1} \quad ; \quad (a),$$

as may be seen from the definition of these symbols.

Superimposing this relationship on formulas (0), relationship (1) results. (1)

Formula (1) may now be used for some specific values of n and i to obtain a whole array of otherwise unrelated facts: a line segment has two points as its extremities ($M_{1,0}=2$); a square has four vertices ($M_{2,0}=4$) and four edges ($M_{2,1}=4$), and a cube has eight vertices ($M_{3,0}=8$), twelve edges ($M_{3,1}=12$) and six faces ($M_{3,2}=6$).

2. Relationships among these manifolds:

An horizontal summation in Table I results in:

$$\sum_{i=o}^{n} M_{n,i} = 3^n \qquad (2)$$

Proof:

Explicitly stated, relationship (2) shows that

$$M_{n,o} + M_{n,1} + \cdots + M_{n,i} + \cdots + M_{n,n} = 3^n,$$

or—according to (1)—

$$2^n + 2^{n-1}n + \cdots + 2^{n-i}\binom{n}{i} + \cdots + 2\binom{n}{n-1} + 1 = 3^n.$$

However, since the left member of this statement represents the binomial expansion of $(2 + 1)^n$, the conclusion follows immediately.

3. Euler's generalized relationship:

In 3-dimensional space, V, the number of vertices, E, the number of edges and F, the number of faces of any solid are interconnected by the well-known Euler-formula

$$V + F = E + 2$$

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(1) Multiply both sides of (a) by 2^{n-i}, and rewrite the resulting equation as

$$2^{n-i}\binom{n}{i} = 2\cdot 2^{n-1-i}\binom{n-1}{i} + 2^{n-1-(i-1)}\binom{n-1}{i-1}.$$

This, however, becomes interpreted as:

$$M_{n,i} = 2M_{n-1,i} + M_{n-1,i-1}.$$

This law, which—in our terminology—reads

$$M_{3,0} + M_{3,2} = M_{3,1} + 2 ,$$

can be generalized, at least in the case of hyper-cubes. It becomes:

$$\sum_{i=o}^{n} (-1)^i M_{n,i} = 1 \qquad (3).$$

Proof:

In detail, (3) states that

$$M_{n,o} - M_{n,1} + M_{n,2} - M_{n,3} + \cdots + (-1)^i M_{n,i} + \cdots + (-1)^n M_{n,n} = 1.$$

Similar to the above proof for equation (2), this means that

$$2^n - 2^{n-1}n + \cdots + (-1)^i 2^{n-i}\binom{n}{i} + \cdots + (-1)^n = 1.$$

Since the left side represents the binomial expansion of $(2-1)^n$, the proof is established.

4. Lengths of hyper-diagonals:

The relationship

$$D_{n,i} = a\sqrt{i} \qquad (4)$$

where $D_{n,i}$ is the length of the i-dimensional (hyper-) diagonal of an n-dimensional hyper-cube, is immediately obvious. It is stated here for completeness' sake.

5. Hyper-volumes:

From the definition of (hyper-) volumes, it is clear that

$$V_{n,i} = M_{n,i} V_{i,i} = M_{n,i} a^i \qquad (5)$$

With the help of relationship (1), this may also be expressed as

$$V_{n,i} = 2^{n-i}\binom{n}{i}a^i \qquad (5a)$$

Selecting some specific values for *i* and *n*, all of the following become special cases of the above:

A line segment (of length *a*) has 2 vertices ($V_{1,0}=2$). Its length is *a* ($V_{1,1}=a$).

A square (whose side is *a* units long) has 4 vertices ($V_{2,0}=4$).
Its perimeter is 4a ($V_{2,1}=4a$).
Its area is a^2 ($V_{2,2}=a^2$).

A cube (whose side is *a* units long) has 8 vertices ($V_{3,0}=8$).
The total length of its edges is 12a ($V_{3,1}=12a$).
Its surface measures $6a^2$ ($V_{3,2}=6a^2$) and its volume contains a^3 cubic units ($V_{3,3}=a^3$).

6. Circum-and in-scribed hyper-spheres[2]:

[2] A hyper-sphere is defined as the manifold composed of the set of points—in *n* dimensional space—which are equi-distant from a fixed point. The fixed point is the "center of the hyper-sphere", the constant distance is the "hyper-radius."

It is immediately seen that

$$R_n = \frac{D_{n,n}}{2} = \frac{a\sqrt{n}}{2}$$

and $\qquad (6)$

$$r_n = \frac{a}{2}$$

and, thus,

$$\frac{R_n}{r_n} = \sqrt{n} \qquad (6a)$$

where R_n and r_n represent the lengths of the (hyper-) radii of n-dimensional circum- and in- (hyper) spheres [2], respectively.

The lengths of the circum- and in-radius of a square of side *a*), known to be $\frac{a\sqrt{2}}{2}$ and $\frac{a}{2}$, respectively, and the lengths of the radii of spheres circum- and in-scribed about a cube (of side a) as

$\frac{a\sqrt{3}}{2}$, or else $\frac{a}{2}$, are now special cases.

As a corollary of (6a), the ratio between the hyper-volumes of the two respective hyper-spheres is given as

$$\frac{V_{\text{circumscribed hyper-sphere}}}{V_{\text{inscribed hyper-sphere}}} = \sqrt{n}^{\,n} \qquad (6b)$$

This is an immediate consequence of the fact that—if two linear measurements of two (hyper-) solids have ratio r the ratio of their n-dimensional measures is r^n.

The Regular Hyper-Tetrahedron:

Start again with a point, i.e., an element of a point set of dimensionality zero. Take this point out of its space over a distance of *a* units to produce a line segment, a 1-dimensional configuration. It is defined by the original and the new position of the point. Locate the center of this line segment. At this point erect a line perpendicular to the line segment and such that its endpoint be *a* units from the extremities of the given line segment. This forms an equilateral triangle, a 2-dimensional entity, defined again by the two extremities of the line segment and the endpoint of the perpendicular. Find the center of this triangle. At this point draw a perpendicular to the plane of the equilateral triangle and such that its endpoint has a distance of *a* units from the vertices of the given triangle. The 3-dimensional solid created in this manner is a regular tetrahedron, uniquely determined by its vertices. Then obtain the center of this tetrahedron, using it as the footpoint of a line perpendicular to the 3-dimensional

space of the tetrahedron and of such length that its endpoint has a distance of *a* units from the vertices of the regular tetrahedron. So we have a regular simplex, a 4-dimensional hyper-solid, which corresponds to the tesseract in the former sequence of hyper-cubes. This process is repeated indefinitely for the further members of the set of n-dimensional regular hyper-tetrahedrons.

Here, the recurrence formulas

$$M_{n,i} = M_{n-1,i} + M_{n-1,i-1}\ ; \quad \begin{cases} n > 0 \\ n \geq i \end{cases}$$

$$M_{n,o} = M_{n-1,o} + 1 = n + 1; \quad n > 0 \qquad (7)$$

$$M_{o,o} = 1$$

which translate the definition, are used to set up Table II.

Relationships:

1. Number of boundary manifolds:

$$M_{n,i} = \binom{n+1}{i+1} \qquad (8)$$

Proof:

Similar to formula (1), (8) is based on

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1} ,$$

which may also be symbolized by

$$\binom{n+1}{i+1} = \binom{n}{i+1} + \binom{n}{i} .$$

Combine this equality with recurrence formula (7), and statement (8) results. [3]

Choosing specific values again, our theme re-occurs. This one relationship encompasses such facts as pertain to the number of extremities of a line segment, the number of vertices or edges of an equilateral triangle, and the number of vertices, edges and faces of a regular tetrahedron.

2. Relationships among these manifolds:

Applying horizontal summation to rows in Table II, as had similarly been done to Table I, it may be discerned that

$$\sum_{i=o}^{n} M_{n,i} = 2^{n+1} - 1 \qquad (9)$$

Proof:

The left member of (9) reads

$$M_{n,o} + M_{n,1} + M_{n,2} + \cdots + M_{n,i} + \cdots + M_{n,n},$$

or - according to (8) -

$$(n+1) + \binom{n+1}{2} + \binom{n+1}{3} + \quad + \binom{n+1}{i} + \quad + \binom{n+1}{n+1}.$$

But the right member of (9) is

$$(1 + 1)^{n+1} - 1,$$

which - upon using the binomial theorem - becomes:

$$[1 + (n+1) + \binom{n+1}{2} + \cdots + \binom{n+1}{i} + \cdots + \binom{n+1}{n+1}] - 1,$$

and, hence, conclusion (9) follows.

3. Euler's generalized relationship:

$$\sum_{i=o}^{n} (-1)^i M_{n,i} = 1 \qquad (3)$$

holds here also.

Proof:

The left side of equation (3)—utilizing formula (8)—reads:

$$(n + 1) - \binom{n+1}{2} + \binom{n+1}{3} - \cdots + (-1)^{i-1}\binom{n+1}{i} + \cdots + (-1)^n\binom{n+1}{n+1}.$$

Now observe that

$$0 = (1 - 1)^{n+1} = 1 - (n + 1) + \binom{n+1}{2} - \cdots + (-1)^i\binom{n+1}{i} + \cdots + (-1)^{n+1}\binom{n+1}{n+1}.$$

This, however, causes the above expression, the left side of (3), to equal 1.

4. Hyper-heights:

Limit the discussion to n-dimensional hyper-heights of n-dimensional regular hyper-tetrahedrons, symbolizing them by h_n. This does not affect the generalization, since

$$h_{n,i} = h_{i,i}.$$

Then,

$$h_n = \frac{a}{n}\sqrt{\binom{n+1}{2}} = a\sqrt{\frac{n+1}{2n}} \qquad (10)$$

Proof:

On the basis of the definition of these hyper-solids, the following observations may be made:

$$h_1^2 = a^2$$

$$h_2^2 = h_1^2 - \frac{h_1^2}{4} = \frac{a^2}{(2!)^2}(3)(1)$$

$$h_3^2 = h_2^2 - \frac{h_2^2}{9} = \frac{a^2}{(3!)^2}(4\cdot3)(2\cdot1)$$

$$h_4^2 = h_3^2 - \frac{h_3^2}{16} = \frac{a^2}{(4!)^2}(5\cdot4\cdot3)(3\cdot2\cdot1)$$

applying mathematical induction,

$$h_n^2 = h_{n-1}^2 - \frac{h_{n-1}^2}{n^2} = \frac{a^2}{(n!)^2}\frac{(n+1)!}{2}(n-1)! \ [4]$$

or,

$$h_n = \frac{a}{n}\sqrt{\binom{n+1}{2}} ,$$

[3] Interpret $\binom{n+1}{i+1}$ as $M_{n,i}$, $\binom{n}{i+1}$ as $M_{n-1,i}$ and $\binom{n}{i}$ as $M_{n-1,i-1}$.

4. While the iteration only starts with n=2, (h_o is undefined) relationship (10) is valid for all natural numbers n.

and—thus—relationship (10) is justified.

Furthermore, one formula establishes h_2, the height of an equilateral triangle as $\frac{a}{2}\sqrt{3}$, and h_3, the (3-dimensional) height of a regular tetrahedron as $\frac{a}{3}\sqrt{6}$.

5. Hyper-volumes:

First, we show that $V_{n,n}$, the n-dimensional content of our regular hyper-tetrahedron, is given by:

$$V_{n,n} = \frac{a^n}{n!}\sqrt{\frac{n+1}{2^n}} = \frac{\prod_{i=1}^{n} h_i}{n!} \qquad (11)$$

Proof:

For the first equality of (11), an inductive procedure is used. It is evident that

$$V_{1,1} = \frac{1}{1} V_{o,o} h_1 = a$$

$$V_{2,2} = \frac{1}{2} V_{1,1} h_2 = \frac{a^2}{2}\sqrt{\frac{3}{2^2}}$$

$$V_{3,3} = \frac{1}{3} V_{2,2} h_3 = \frac{a^3}{3!}\sqrt{\frac{4}{2^3}}$$

hence,

$$V_{k,k} = \frac{1}{k} V_{k-1,k-1} h_k = \frac{a^k}{k!}\sqrt{\frac{k+1}{2^k}}$$

will be assumed correct. Then,

$$V_{k+1,k+1} = \frac{1}{k+1} V_{k,k} h_{k+1} = \frac{1}{k+1}\frac{a^k}{k!}\sqrt{\frac{k+1}{2^k}}\,\frac{a}{k+1}\sqrt{\left(\frac{k+2}{2}\right)}$$

$$= \frac{a^{k+1}}{(k+1)!}\sqrt{\frac{k+2}{2^{k+1}}}\text{, Q. E. D.}$$

For the second equational part of (11), use relationship (10) to form

$$\frac{1}{n!}\prod_{i=1}^{n} h_i .$$

It becomes:

$$\frac{a}{(n!)^2}\sqrt{\frac{1\cdot 2}{2}\,\frac{2\cdot 3}{2}\,\frac{3\cdot 4}{2}\cdots\frac{(n-1)n}{2}\,\frac{n(n+1)}{2}} = \frac{a^n}{n!}\sqrt{\frac{n+1}{2^n}} .$$

which equals $V_{n,n}$.

Furthermore, $V_{n,i}$, the i-dimensional (hyper-) content of our n-dimensional tetrahedron obeys the law that:

$$V_{n,i} = M_{n,i}V_{i,i} = M_{n,i}\frac{a^i}{i!}\sqrt{\frac{i+1}{2^i}} = \binom{n+1}{i+1}\frac{a^i}{i!}\sqrt{\frac{i+1}{2^i}} \qquad (12)$$

Proof:

The first part of this continued equation utilizes the definition of hyper-contents. The remaining equalities follow by substitution from (11) and (8).

Once again, we realize the wealth of information contained in (12). It states—among many other things—that our original line segment has 2 vertices and length *a* ($V_{1,0}=2$, $V_{1,1}=a$), that our equilateral triangle has 3 vertices, perimeter 3a and area $\frac{a^2}{4}\sqrt{3}$ ($V_{2,0}$, $V_{2,1}$, and $V_{2,2}$, respectively.) Our regular tetrahedron has 4 vertices, 6a as the total length of its edges, a surface of $a^2\sqrt{3}$ and a volume of $\frac{a^3\sqrt{2}}{12}$ (by computing $V_{3,i}$ for i = 0, 1, 2, 3.)

6. Relationship between hyper-volumes and hyper-heights:

It may be interesting to note that

$$V_{n,n} = \frac{1}{n} V_{n-1,n-1} h_n \qquad (11a)$$

which combines some previously established results.

7. Circum- and inscribed hyper-spheres:

(a) Relationship between hyper-radii and hyper-heights:

$$R_n = \frac{n}{n+1} h_n$$

$$r_n = \frac{1}{n+1} h_n \qquad (13),$$

where R_n and r_n symbolize the hyper-radii of the n-dimensional circum- and inscribed hyper-spheres, respectively. Thus,

$$R_n + r_n = h_n \qquad (13a)$$

Proof:

(13) may be proved by mathematical induction, starting from the initial observation that $r_2 = \frac{h_2}{3}$ and $r_3 = \frac{h_3}{4}$. In like manner, R_n is found.

(b) Relationship between hyper-radii and sides:

$$R_n = a\sqrt{\frac{n}{2(n+1)}}$$

$$r_n = \frac{a}{\sqrt{2n(n+1)}} \qquad (14)$$

and

$$\frac{R_n}{r_n} = n \qquad (14a)$$

Proof:

(14) may be recognized quite readily by using some of the earlier relationships ((13) and (10)). (14a) follows from combining both forms of (14).

This generalized relationship enables us to obtain the radii of the circum- and in-circle of our equilateral triangle,

$$R_2 = \frac{a\sqrt{3}}{3} \text{ and } r_2 = \frac{a\sqrt{3}}{6},$$

as well as the radii of circum- and inscribed spheres of our tetrahedron as $R_3 = \frac{a\sqrt{6}}{4}$ and $r_3 = \frac{a\sqrt{6}}{12}$.

Corollary:

The ratio between the volumes of the two respective hyper-spheres is given by

$$\frac{V_{\text{circumscribed hyper-sphere}}}{V_{\text{inscribed hyper-sphere}}} = n^n \qquad (14b)$$

Proof:

The reasoning is analogous to the proof for (6b). This points up a very rapid increase in this ratio for higher and higher dimensions. Thus, although the area of a circle circumscribed about an equilateral triangle is only 4 times as large as the area of its in-circle, the hyper-volume of an hyper-sphere circumscribed about a regular simplex is 256 times as large as the hyper-volume of the in-scribed hyper-sphere. [5]

TABLE I

Mu,i=the number of i-dimensional boundary manifolds of an n-dimensional hyper-cube; i=n.

NAME	n \ i	0	1	2	3	4	5	6	7	8	9	10	11	12	--
POINT	0	1													
LINE-SEGMENT	1	2	1												
SQUARE	2	4	4	1											
CUBE	3	8	12	6	1										
TESSERACT	4	16	32	24	8	1									
5-DIM. HYPER-CUBE	5	32	80	80	40	10	1								
6-DIM. "	6	64	192	240	160	60	12	1							
7-DIM. "	7	128	448	672	560	280	84	14	1						
8-DIM. "	8	256	1024	1792	1792	1120	448	112	16	1					
9-DIM. "	9	512	2304	4608	5376	4032	2016	672	144	18	1				
10-DIM. "	10	1024	5120	11520	15360	13440	8064	3360	960	180	20	1			
11-DIM. "	11	2048	11264	28160	42240	42240	29568	14784	5280	1320	220	22	1		
12-DIM. "	12	4096	24576	67584	112640	126720	101376	59136	25344	7920	1760	264	24	1	
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TABLE II

Mu,i=the number of i-dimensional boundary manifolds of an n-dimensional regular hyper-tetrahedron; i=n.

NAME	n \ i	0	1	2	3	4	5	6	7	8	9	10	11	12	---
POINT	0	1													
LINE-SEGMENT	1	2	1												
EQUILATERAL TRIANGLE	2	3	3	1											
REGULAR TETRAHEDRON	3	4	6	4	1										
" SIMPLEX	4	5	10	10	5	1									
REG. 5-DIM. HYPER-TETRAHEDR.	5	6	15	20	15	6	1								
" 6- " "	6	7	21	35	35	21	7	1							
" 7- " "	7	8	28	56	70	56	28	8	1						
" 8- " "	8	9	36	84	126	126	84	36	9	1					
" 9- " "	9	10	45	120	210	252	210	120	45	10	1				
" 10- " "	10	11	55	165	330	462	462	330	165	55	11	1			
" 11- " "	11	12	66	220	495	792	924	792	495	220	66	12	1		
" 12- " "	12	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1	
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(5) The verification of this statement, however, exceeds the scope of this paper. It entails the relationship for the hyper-volume of an n-dimensional hyper-sphere. $V_n = \frac{\sqrt{\pi^n}}{\Gamma(\frac{n}{2}+1)} r^n$