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BEAMS RESTING ON IDEALIZED SOIL MODEL

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ABSTRACT

Flexural behavior of beams resting on a homogenous elastic soil medium is examined. In this soil-foundation-interaction analysis, the soil is idealized as a Winkler medium. Beams with free ends which are subjected to uniform and concentrated loads and concentrated moments are analyzed. The overall properties of beam and soil medium is expressed in terms of a dimensionless characteristic parameter termed as relative rigidity.

From a computer program, numerical results are developed to illustrate the influence of the relative rigidity of a beam on its deflections, flexural moments and shear forces.

INTRODUCTION

Beams supported on deformable elastic media constitute problems of importance in the analysis and design of structural foundations resting on rock and soil media (Hetenyi, 1946; Seely and Smith, 1952; Selvadurai, 1979). Examples of beams resting on soil media include the behavior of rails and rail-road ties, grillage and distributor beams in a floor system. Since the fundamental laws of stresses and strains for soils are quite complex, mathematically simple idealizations of soil behavior are often utilized for the analytical study of soil-structure–interaction problems. One such model commonly used to represent the soil medium is Winkler (Spring) Model. This model appears to have been first utilized by E. Winkler in 1867. The exact analytical solution for a prismatic beam resting on a Winkler medium was obtained by Hetenyi (1946). It involves the solution of a fourth order differential equation which requires rigorous analytical treatments when applied to most practical situations.

The objective of this study is to develop an elastic soil-foundation-interaction analysis for prismatic beams resting on a homogenous Winkler medium. Only beams subjected to uniform and concentrated loads and concentrated moments are considered. The overall properties of beam-foundation system are completely defined by relative rigidity. A computer program is developed to analyze the system in terms of flexural and deformational characteristics at one end of beam. Numerical results are evaluated to illustrate the effect of relative rigidity on the behavior of beams. Influence lines for deflection, slope, moment and shear functions due to various loads on rigid and flexible beams covering a wide range of relative rigidity have also been drawn.

Winkler's Model

The Winkler's Model assumes that the deflection (y) at any point on the surface of an idealized elastic foundation is directly proportional to the pressure (p) at that point, and completely independent of pressures or deflections occurring at other immediately neighboring points along the length of the beam or the foundation. In other words:

$$p = ky \tag{1}$$

where k is constant. The constant k is described as the modulus of subgrade reaction. In practice, the subgrade modulus for a given soil can be determined by various methods (Terzaghi, 1956; Miner and Seastone, 1955; Selvadurai, 1979). Winkler's model formed the basis of H. Zimmerman's classical work in the analysis of railroad track published in 1884. Hetenyi (1946) reviewed the analysis of floor-systems for ships, buildings and floating bridges, boilers, pressure vessels and containers as well as large-span modern reinforced concrete halls and domes. Hertz's (1884) analysis of a floating circular plate is also based on this assumption.

While Winkler's theory holds rigorously for most of the problems mentioned here, its application to soil-structure interaction problems should be regarded only as a practical approximation (Hetenyi, 1946). In spite of its simplicity, this theory is widely used. In a review paper, Hetenyi (1966) catalogued various works related to interaction analysis. It was shown that, by far the largest number of investigations in this field are based on Winkler's hypothesis. Selvadurai (1979) concludes that for flexible beams and plates that have high relative rigidity, Winkler's model provides a satisfactory representation of the deformational characteristics of the soil medium.

EXACT ANALYSIS

Consider a straight beam of constant width B and length L and supported along its entire length by a

Winkler type of foundation. Assume that E and I are respectively the modulus of elasticity and the moment of inertia of the beam. If the beam is subjected to a uniformly distributed load as shown in Figure 1, then the differential equation governing its deflected shape is given by:

$$EI\frac{d^4y}{dx^4} + Bky - q = 0 \qquad (2)$$

The general solution for Equation 2 is given by Hetenyi (1946) which is expressed in terms of certain complex hyperbolic functions of beam and soil properties (see box on this page). Such a solution also uses the displacement, rotation, moment and shear forces at the free end of the beam $x = 0$. This analytical solution has been computerized so that displacement, rotation, shear and moment due to uniform and concentrated loads or concentrated moments at any point on the beam could be determined.

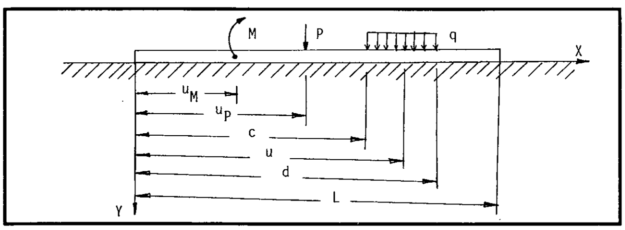

Figure 1. Typical loads on a beam resting on Winkler Medium.

RESULTS

To illustrate the effect of relative rigidity on a beam, several influence lines have been drawn. A set of results obtained from the analysis of a beam with a single concentrated load is presented in Figure 2. It may be observed that when $\lambda L > 4$, the displacement at or near the edges of the beam becomes negative. This is justified by the assumed idealized connection between the foundation and soil.

If the entire span of an infinite beam is acted on by a uniformly distributed load, then the displacement at every point of the beam will be constant. The moment and shear at every point on the beam will be equal to zero. Such behavior will occur regardless of the rigidity of the beam. To study the effect of uniform loads spread over a portion of a beam, the problem as shown in Figure 3 is examined for a wide range of relative rigidities. It may be of interest to note the variation in beam displacements as λL increases. For $\lambda L = 10$, the displacement of the beam remains almost limited to the loaded portion. With further increase in λL, this effect becomes pronounced and at $\lambda L = 50$, the distribution of displacements below the load approaches near uniform. This is consistent with the Winkler's hypothesis. As $\lambda L \rightarrow \infty$, this hypothesis indicates a beam of such flexibility that the load could be considered to have been placed directly on the soil. In that case, uniform displacements should occur below the load. Since $\lambda L = 50$ approximates such a situation, the displacement profile different from a uniform one (as in Figure 3) is justified. Due to uniform load, the flexural moment and the shear force in the beam diminish with the increase in λL. Since the displacements approach uniformity at high values of λL, almost no flexural moment and shear force should develop. This fact is substantiated in Figure 4. The effect of a single concentrated moment on the deflection of a beam is represented in Figure 5. It may be observed that for $\xi = 0.5$, the displacement at the center of the beam is zero. This condition is in agreement with the analytical solution obtained by Hetenyi (1946).

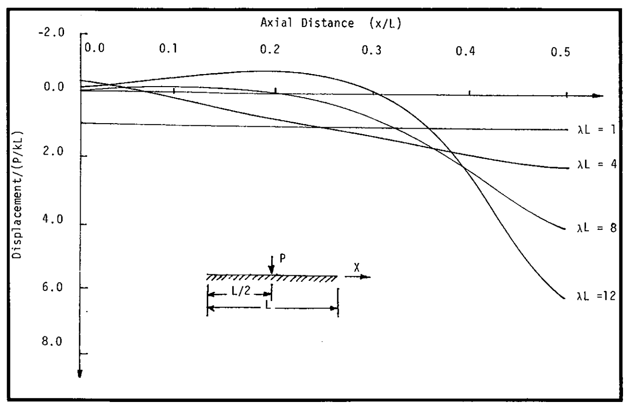

Figure 2. Effect of relative rigidity on displacements of a beam

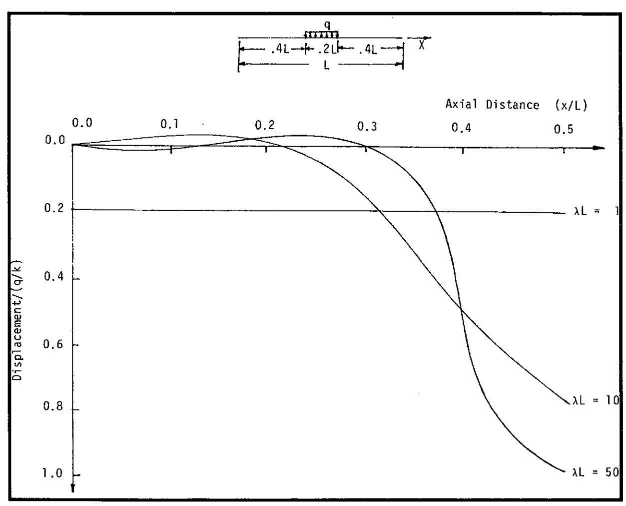

Figure 3. Effect of relative rigidity (λL) on beam displacements.

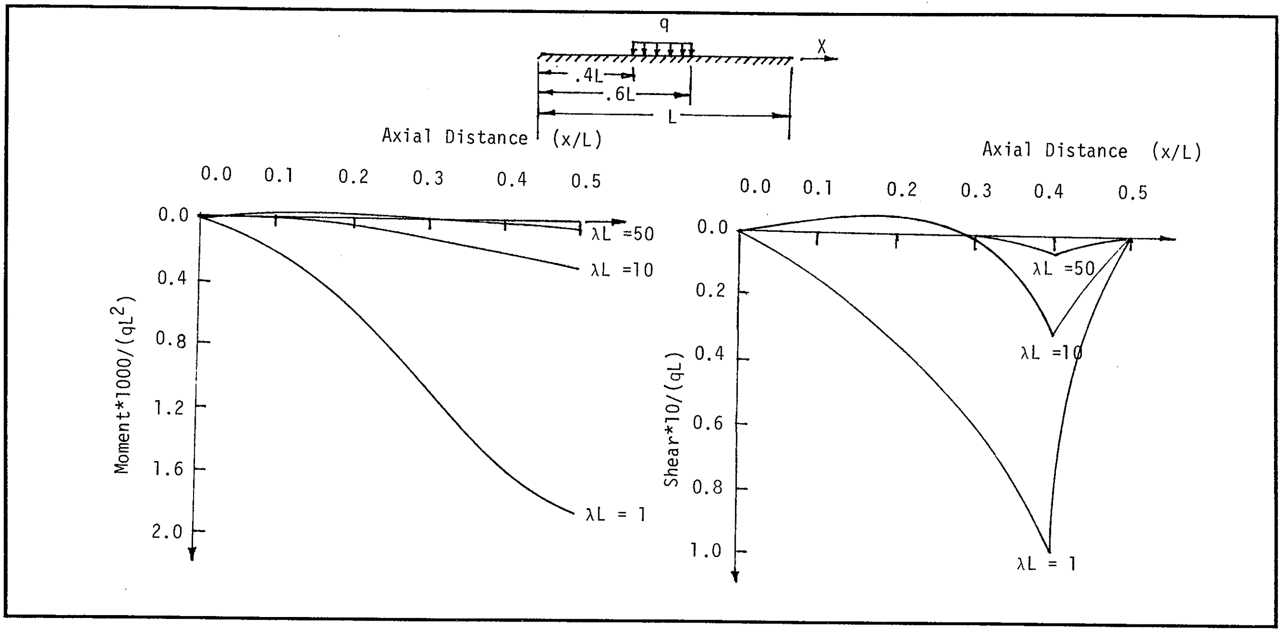

Figure 4a. Effect of λL on moment

Figure 4b. Effect of λL on shear.

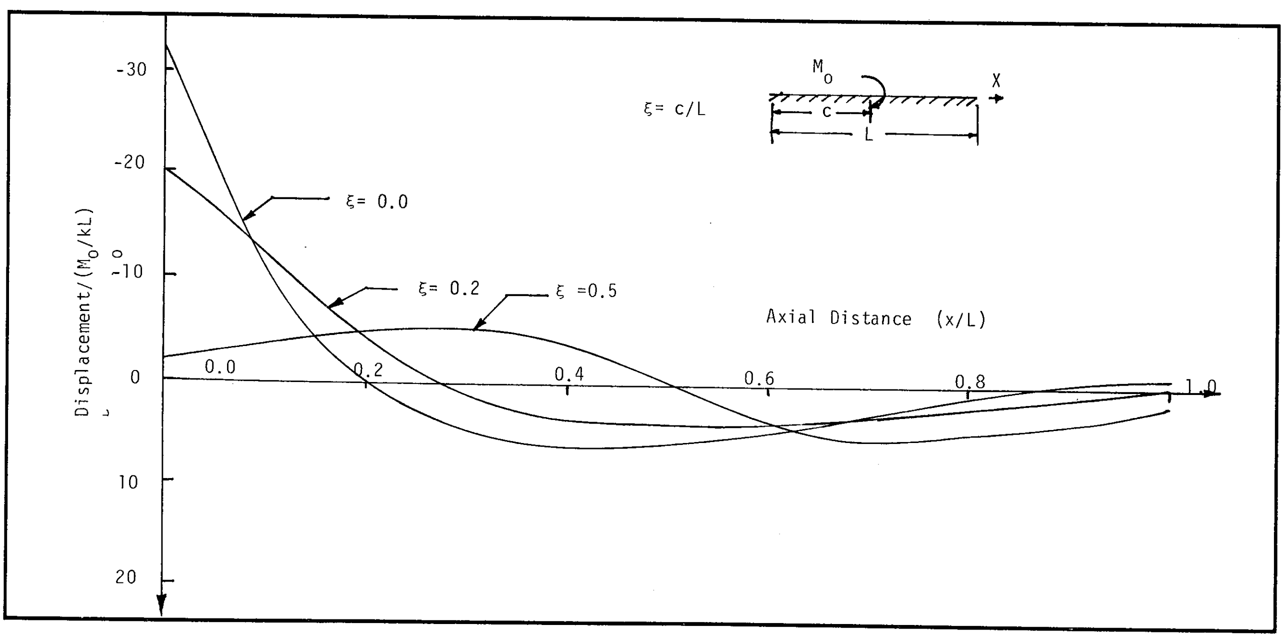

Figure 5. Variation of displacements with respect to the position (ξ) of load — based on $\lambda L = 4$.

General Solution of the Beam Equation

$$y_x = y_0F_1(\lambda x) + \frac{1}{\lambda}\theta_0F_2(\lambda x) - \frac{1}{\lambda^2EI}M_0F_3(\lambda x) - \frac{V_0}{\lambda^3EI}F_4(\lambda x) + \frac{1}{\lambda^3EI}PF_4[\lambda(x-u_P)] + \frac{1}{\lambda^3EI}\int_c^x qF_4[\lambda(x-u)]\,du - \frac{1}{\lambda^2EI}MF_3[\lambda(x-u_M)]$$

$$\theta_x = -4\lambda y_0F_4(\lambda x) + \theta_0F_1(\lambda x) - \frac{1}{\lambda EI}M_0F_2(\lambda x) - \frac{1}{\lambda^2EI}V_0F_3(\lambda x) + \frac{1}{\lambda^2EI}PF_3[\lambda(x-u_P)] + \frac{1}{\lambda^2EI}\int_c^x qF_3[\lambda(x-u)]\,du - \frac{1}{\lambda EI}MF_2[\lambda(x-u_M)]$$

$$M_x = \frac{Bk}{\lambda^2}y_0F_3(\lambda x) + \frac{Bk}{\lambda^3}\theta_0F_4(\lambda x) + M_0F_1(\lambda x) + \frac{1}{\lambda}V_0F_2(\lambda x) - \frac{1}{\lambda}PF_2[\lambda(x-u_P)] - \frac{1}{\lambda}\int_c^x qF_2[\lambda(x-u)]\,du + MF_1[\lambda(x-u_M)]$$

$$V_x = \frac{Bk}{\lambda}y_0F_2(\lambda x) + \frac{Bk}{\lambda^2}\theta_0F_3(\lambda x) - 4\lambda M_0F_4(\lambda x) + V_0F_1(\lambda x) - PF_1[\lambda(x-u_P)] - \int_c^x qF_1[\lambda(x-u)]\,du - 4\lambda MF_4[\lambda(x-u_M)]$$

where y_0, θ_0, M_0 and V_0 are respectively the displacement, rotation, moment and shear at the end of the beam at $x = 0$. Other symbols are defined in Figure 1. Also:

$$\lambda L = \left(\frac{BkL^4}{4EI}\right)^{1/4}$$

$$F_1(\lambda x) = \cosh\lambda x\cos\lambda x$$

$$F_2(\lambda x) = \frac{1}{2}(\cosh\lambda x\sin\lambda x + \sinh\lambda x\cos\lambda x)$$

$$F_3(\lambda x) = \frac{1}{2}\sinh\lambda x\sin\lambda x$$

$$F_4(\lambda x) = \frac{1}{4}(\cosh\lambda x\sin\lambda x - \sinh\lambda x\cos\lambda x)$$

CONCLUSIONS AND RECOMMENDATIONS

From this study it is concluded that the behavior of a beam resting on Winkler media can be predicted by the characteristic factor λL which incorporates the beam and soil properties. Influence lines for different functions can also profitably be used in the analysis and design of foundations. For beams with higher λL, flexible behavior is predicted so that the beam-deflection is localized at or near the point of load application. Rigid beams have lower values of λL; they undergo uniform displacement with higher values of moment and shear than those for flexible beams. This study has indicated the need for further research in many areas. Only elastic prismatic beams having free edges have been considered. Further studies could be extended to inelastic range with non-prismatic beams under any support and loading conditions.

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